Preprojective algebras and fractional Calabi-Yau algebras

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Plan

- What do the words mean?
- What's the rough argument?
- How do you do it properly?
- Higher homological algebra

Reference:

- "*Serre functors and graded categories*" (`2007.01817`)
- also: "*The Nakayama automorphism of a self-injective preprojective algebra*", Bull. LMS 2020, (`1906.11817`)

What do the words mean?

Preprojective algebras and fractional Calabi-Yau algebras

Given a quiver we consider two algebras: its path algebra and its preprojective algebra. If the quiver is Dynkin (ADE) then both have nice properties: the path algebra is fractionally Calabi-Yau and the preprojective algebra has a Nakayama automorphism of finite order. I will explain what these words mean and how these properties are related, using 2-dimensional category theory. This gives a useful criterion to check if a *d*-representation finite algebra is fractionally Calabi-Yau.

Preprojective algebra (1)

Given a quiver, double it and impose "fake commutativity" relations.

$$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3, \qquad \overline{Q} = 1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3$$

$$aa^* = b^*b, \quad a^*a = 0, \quad bb^* = 0$$

If underlying graph is Dynkin (ADE), get a finite dimensional algebra Π. Study its projective and injective representations.

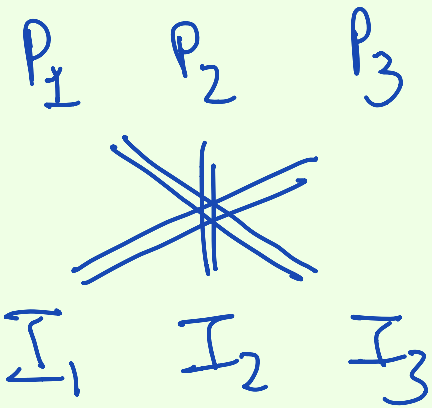
$P_1 =$	1	2	3		3	2	1
	2	1 3	2		2	1 3	2
	3	2	1	$I_1 =$	1	2	3

projectives — injectives

Preprojective algebra (2)

They're the same! (Up to a permutation.)

This is because Π is self-injective/Frobenius. Note that $\alpha^2 = 1$.



$${}_{\Pi}\Pi_{\Pi} \cong {}_{\Pi}\Pi^{*}_{\alpha}$$

Nakayama aut.

ρ: $1 \leftrightarrow 3$, $2 \circlearrowleft$

$$\alpha(e_i) = e_{\rho(i)}.$$

This was "folklore" and proved by [Brenner-Butler-King 2002].

Path algebra (1)

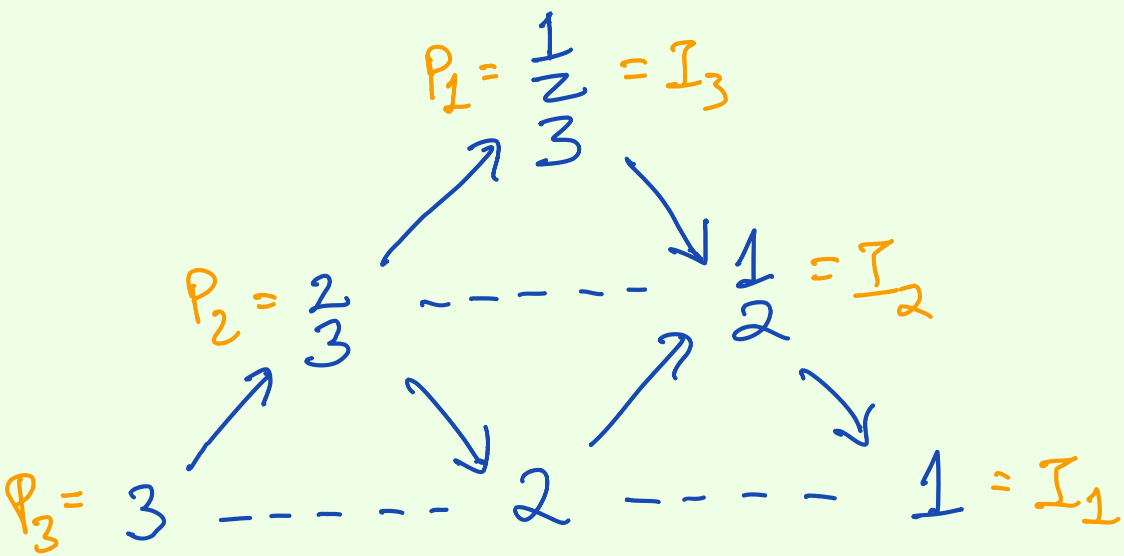
$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$

$kQ = \langle e_1, e_2, e_3, a, b, ba \rangle$

Gabriel's Theorem:

Q is Dynkin ⇔ *kQ* has finitely many indecomposable modules.

We can draw the category ind(*kQ*) of indecomposable modules. This picture is called the Auslander-Reiten quiver.



Path algebra (2)

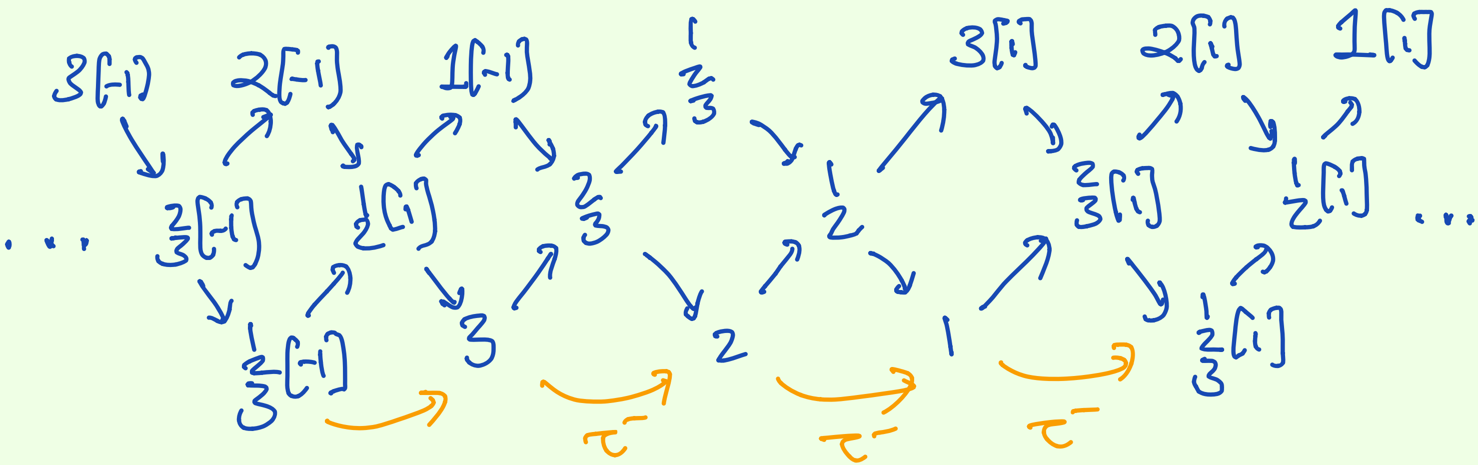
The derived category $D^b(kQ)$ has indecomposable objects

$\mathbb{Z} \times \mathrm{ind}(kQ)$.

We can draw its picture (AR quiver) too.

$\Sigma M = M[1]$

Note the shift functor [1], also written Σ.



Serre functor (1)

Let *C* be a linear category (its hom sets are vector spaces).

A functor *S*: *C* → *C* is called a *Serre functor* if it satisfies Serre duality:

$$\mathcal{C}(x,y) \xrightarrow{\sim} \mathcal{C}(y, Sx)^* \text{ natural in } x, y \in \mathcal{C}$$

As *kQ* has finite (global and vector space) dimension, $D^b(kQ)$ has a Serre functor. It sends projectives to injectives.

C is *fractionally Calabi-Yau* if ∃ *p*, *q* ∈ ℤ, and a relation

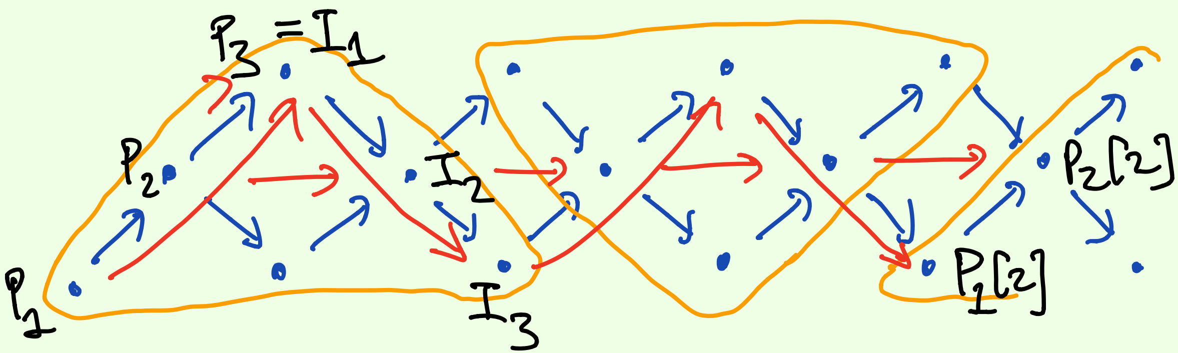
$$S^q \cong \Sigma^p$$

Serre functor (2)

Q: ADE Dynkin.

$D^b(kQ)$ is fractionally Calabi-Yau: $S^{p+2} \cong \Sigma^p$

This was "folklore" and proved by [Miyachi-Yekutieli 2001].



$S^4 \cong \Sigma^2$.

The BBK and MY results are known to be related in some cases [Herschend-Iyama 2011a].

We want a general result: detect fCY via Nakayama autom.

What's the rough argument? (1)

S and Σ commute. So the fractional Calabi-Yau relation $S^{p+2} = \Sigma^p$ can be rearranged:

$$S^{p+2} = \Sigma^p$$
$$S^{p+2} = (S^p S^{-p})\Sigma^p$$
$$S^2 = (S^-\Sigma)^p$$
$$= \tau^{-p}$$

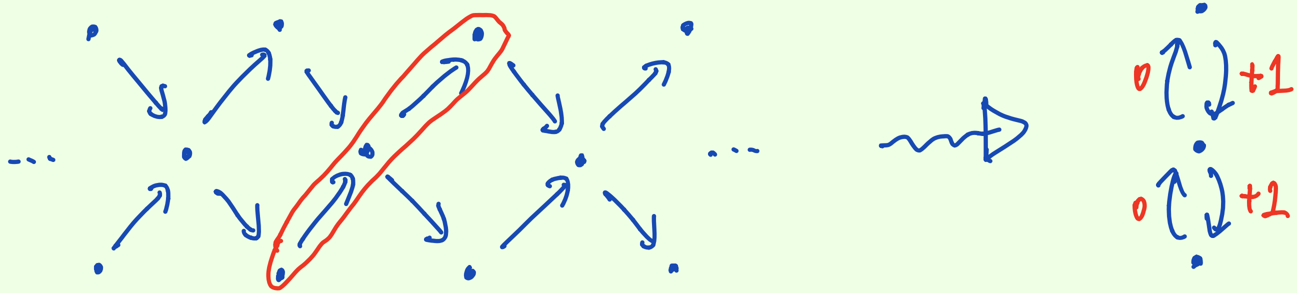
use $S^-\Sigma = \Sigma S^-$

$(S^{-1}\Sigma) = \tau^-$, the (derived inverse) AR translate. So:

$$S^{p+2} = \Sigma^p \text{ on } \mathrm{D}^b(kQ) \iff S^2 = \tau^{-p} \text{ on } \mathrm{D}^b(kQ).$$

What's the rough argument? (2)

Use orbit category $\mathrm{D^b}(kQ)/\tau^-$. Action of τ^- gives it a grading.



The existence of S on $\mathrm{D^b}(kQ)$ shows Π is self-injective [Iyama-Oppermann, 2013]. With the grading:

$$S^2 = \tau^{-p} \text{ on } \mathrm{D^b}(kQ) \iff S^2 = \mathrm{id}\{p\} \text{ on } \mathrm{D^b}(kQ)/\tau^-$$

What's the rough argument? (3)

A one-object category $C = \{\bullet\}$ defines an algebra $A = C(\bullet,\bullet)$.

$$A = \mathcal{C}(\bullet,\bullet) \cong \mathcal{C}(\bullet, S\bullet)^* = A^*_\alpha$$

A Serre functor S for C gives a Nakayama functor α for A.

Summary:

$S^{p+2} = \Sigma^p$ on $D^b(kQ)$ $\Leftrightarrow$ $S^2 = \tau^{-p}$ on $D^b(kQ)$

"fCY"

$\Leftrightarrow$ $S^2 = \{p\}$ on orbit category

$\Leftrightarrow$ $\alpha^2 = \{p\}$ on Π

"finite order Nakayama"

How do you do it properly?

First, what's the difficulty? $S^{p+2} = \Sigma^p$ should be $S^{p+2} \cong \Sigma^p$.

We have a category, functors, and natural isomorphisms.

This is 2-categorical.

0-cells 1-cells 2-cells

What about the Frobenius algebra? Is this 2-categorical? Yes.

Nakayama autom. is only unique up to inner automorphism.

2-category: algebras, homomorphisms, and inner autom.s.

0-cells 1-cells 2-cells.

Relationship between algebras and categories is 2-functorial.

(on "core").

Orbit categories (1)

Taking orbit categories is a biequivalence [Asashiba 2017]:

Equivariant categories	**Hom-graded categories**
0-cells: (D, F: $D \to D$)	0: C with graded hom spaces
1-cells: (Φ:$D \to D$, ϕ:$\Phi F \to F\Phi$)	1: (H:$C \to C$, γ: degree adjuster)
2-cells: commuting nat. tx.s	2: homogeneous nat. tx.s

Triangulated functors on $D^b(kQ)$ are 1-cells on ($D^b(kQ)$, Σ).

$x \to y \to z \to \Sigma x$ $\qquad$ $\Phi x \to \Phi y \to \Phi z \dashrightarrow \Sigma \Phi x$

$\Phi z \to \Phi \Sigma x \to \Sigma \Phi x$

$\Phi : D^b(kQ) \to D^b(kQ)$, $\quad \Phi\Sigma \xrightarrow{\sim} \Sigma\Phi$

Orbit categories (2)

Strong fractionally Calabi-Yau definition [Keller 2008]:

There exists isom. of equivariant functors on ($\mathrm{D}^{\mathrm{b}}(kQ)$, Σ)

$$(S, s)^p \cong (\Sigma, -1)^q$$

where (S, s) satisfies compatibility condition.

Equivalently, (S, s) is triangulated [Van-den-Bergh 2011].

So far, everything is Σ-equivariant.

But we want to take orbit category by τ^-.

We need to make everything τ^--equivariant.

Change of action (1)

A Serre functor *S* commutes with everything: $F : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$

$$\mathcal{C}(x, SFy) \simeq \mathcal{C}(Fy, x)^* \simeq \mathcal{C}(y, F^{-1}x)^* \simeq \mathcal{C}(F^{-1}x, Sy) \simeq \mathcal{C}(x, FSy)$$

$$\zeta : SF \xrightarrow{\sim} FS$$

So we can make Σ-equivariant functors τ^{-}-equivariant:

$$\Phi \tau^{-} = \Phi S^{-} \Sigma \dashrightarrow S^{-} \Sigma \Phi = \tau^{-} \Sigma$$

$$\Phi S^{-} \Sigma \xrightarrow{\zeta^{-} \Sigma} S^{-} \Phi \Sigma \xrightarrow{S^{-} \phi} S^{-} \Sigma \Phi$$

Change of action (2)

Understand the commutation morphisms well [Keller, Dugas 2012, Chen 2017], so get equivalence of monoidal categories:

$$\mathrm{End}(\mathrm{D^b}(kQ), \Sigma) \cong \mathrm{End}(\mathrm{D^b}(kQ), \tau^{-})$$

$$(S, s)^{-1}(\Sigma, -1) \longmapsto (\tau^{-}, 1)$$

Theorem:

"strong fCY" for $\mathrm{D^b}(kQ)$ $\longmapsto$ $\alpha^2 = \{p\}$ on Π.

Note: α is not classical Nakayama automorphism.

It differs by a sign $(-1)^n$.

Higher homological algebra (1)

Nice properties of *kQ* and Π generalise to algebras Λ with a *d*-cluster tilting module [Iyama 2007].

“*d*-representation finite algebras” ⊂ “d-hereditary algebras”

Our theorem works in this generality:

- $D^b(\Lambda)$ is fractionally Calabi-Yau ⇔ Π has "finite" (graded) Nakayama automorphism.

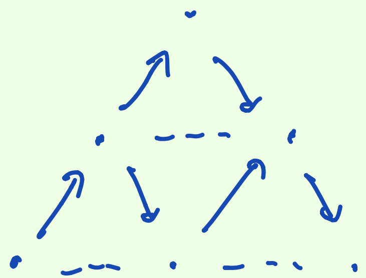
Higher homological algebra (2)

Example: higher Auslander algebras of type A [Iyama 2011]. Both properties are known:

- Nakayama automorphism of Π [Herschend-Iyama 2011a].
- $D^b(\Lambda)$ is frac. Calabi-Yau [Dyckerhoff-Jasso-Walde 2019].

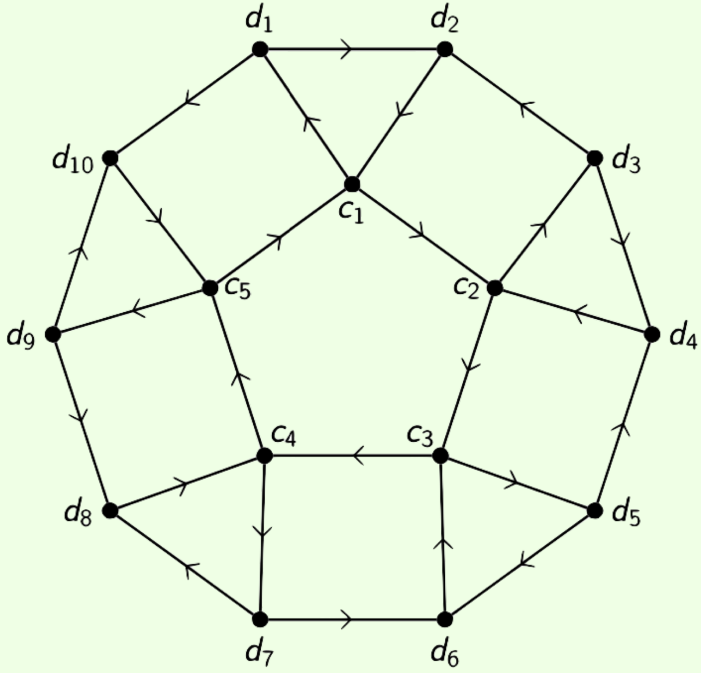
e.g., $\Lambda = kQ/I$

$Q =$



Higher homological algebra (3)

Example: Planar quivers with potential from Postnikov diagrams. These have 2-cluster tilting modules.



When Frobenius, Nakayama automorphism given by diagram rotation [Pasquali 2020].

So taking cuts [Herschend-Iyama 2011b] gives fractional Calabi-Yau algebras.

Thanks for listening!